Transformations of Spinors

- The transformations that commute with $\hat{J}^2$ are unitary transformations with determinant $+1$.
- These are 2×2 matrices **U** such that $\mathbf{U}\mathbf{U}^\dagger = \mathbf{1}$ and $\det \mathbf{U} = +1$. Here $\mathbf{U}^\dagger$ is the *adjoint* of **U**: the *complex conjugate transpose*.
- Such transformations form a group denoted $SU(2)$, the *special (or unimodular) unitary group* in 2-dimensional space.
- The irreps of $SU(2)$ are of order $1, 2, 3 \ldots$: singlet, doublet, triplet. . . .
- This can be written as $2j + 1$ where j runs over nonnegative integers and *half-integers* (i.e., half an odd integer).

Parametrization of $SU(2)$

- A unimodular unitary matrix **U** can be written

$$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

where $aa^* + bb^* = |a|^2 + |b|^2 = 1$. a and b are called the *Cayley-Klein* parameters.

- An alternative, which we will use here, is to write

$$\mathbf{U} = \alpha_0\sigma_0 \; - i(\alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3)$$

with *real* parameters α_p and $\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$, and σ_p are the Pauli matrices.

- These are the *Euler-Rodrigues* parameters.

Parametrization of Rotations (again)

- In terms of the Euler-Rodrigues parameters **U** has the form

$$\begin{pmatrix} \alpha_0 - i\alpha_3 & -i\alpha_1 - \alpha_2 \\ -i\alpha_1 + \alpha_2 & \alpha_0 + i\alpha_3 \end{pmatrix}.$$

- There are three independent real parameters here. This was the same situation we had with the orthogonal matrices comprising $SO(3)$. What is the connection between $SU(2)$ and $SO(3)$?

- In other words, can we use the Euler-Rodrigues parameters to construct rotations in 3-space?

Parametrization of Rotations

- We can generate an orthogonal matrix **O** as follows:

$$\begin{aligned}
O_{11} &= \alpha_0^2 - \alpha_3^2 + \alpha_1^2 - \alpha_2^2 \\
O_{21} &= 2\alpha_0\alpha_3 - 2\alpha_1\alpha_2 \\
O_{31} &= -2\alpha_0\alpha_2 - 2\alpha_1\alpha_3
\end{aligned}$$

$$\begin{aligned}
O_{12} &= -2\alpha_0\alpha_3 - 2\alpha_1\alpha_2 \\
O_{22} &= \alpha_0^2 - \alpha_3^2 - \alpha_1^2 + \alpha_2^2 \\
O_{32} &= -2\alpha_0\alpha_1 + 2\alpha_2\alpha_3
\end{aligned}$$

Parametrization of Rotations

- And finally

$$\begin{aligned} O_{13} &= 2\alpha_0\alpha_2 - 2\alpha_1\alpha_3 \\ O_{23} &= 2\alpha_0\alpha_1 - 2\alpha_2\alpha_3 \\ O_{33} &= \alpha_0^2 + \alpha_3^2 - \alpha_1^2 - \alpha_2^2 \end{aligned}$$

- Since the Euler-Rodrigues parameters are real, all elements of **O** are real. It is easy to show that $\mathbf{O}\mathbf{O}^T = \mathbf{1}$ and $\det \mathbf{O} = +1$.
- Sidebar: various values of E-R parameters and the rotations they generate.

Connection between $SU(2)$ and $SO(3)$

- There is clearly a mapping from $SU(2)$ to $SO(3)$. What is the nature of this connection?
- It can be shown that every matrix in $SU(2)$ gives rise to a (not necessarily unique) matrix in $SO(3)$, and that every matrix in $SO(3)$ can be obtained from a matrix in $SU(2)$.
- But this is is not an isomorphism (one-to-one correspondence).
- Consider two matrices from $SU(2)$, namely, $+\mathbf{1}_2$ and $-\mathbf{1}_2$. In other words, $\alpha_0 = \pm 1$ respectively and the other parameters are zero.

Connection between $SU(2)$ and $SO(3)$

- Both $+\mathbf{1}_2$ and $-\mathbf{1}_2$ generate $+\mathbf{1}_3$ in $SO(3)$!
- For *any* **U** $\in SU(2)$, **U** and $-$**U** generate the *same* orthogonal rotation matrix in $SO(3)$.
- This is a two-to-one *homomorphism* of $SU(2)$ onto $SO(3)$.
- Every irrep of $SU(2)$ can be mapped to an irrep of $SO(3)$.

Odd-dimensional Irreps of $SU(2)$

- From an odd-dimensional irrep of $SU(2)$ we obtain one of the previously discussed irreps of $SO(3)$.

E	∞C_φ	$\dots$
$\vdots$		
$2j+1$	$\frac{\sin(j+\frac{1}{2})\varphi}{\sin\frac{1}{2}\varphi}$	$\dots$
$\vdots$		

- Note that the character of the rotation here is $2j+1$ when $\varphi = 2\pi n$.

Even-dimensional Irreps of $SU(2)$

- From an even-dimensional irrep of $SU(2)$ we obtain a new irrep of $SO(3)$.

E	∞C_φ	$\ldots$
$\vdots$		
$2j+1$	$\frac{\sin(j+\frac{1}{2})\varphi}{\sin\frac{1}{2}\varphi}$	$\ldots$
$\vdots$		

- The degeneracy of the irrep is $2j+1$. Look at what happens when $\varphi = 2\pi$?

Even-dimensional Irreps of $SU(2)$

- The irreps for half-integer j are *double valued*.
- For these irreps, rotation through 2π is not the identity! Only rotation through a multiple of 4π is the identity
- For the double-valued irreps, we have only

  $$D(R)D(S) = \pm D(RS).$$

  This ambiguity cannot be resolved. To Wigner these are “not representations at all”.
- The above relation leads to the theory of “ray representations”.
- Sidebar: example of $D^{\frac{1}{2}}$