

CHEM 190 and CHEM 290

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Homework No. 4 - SAMPLE MIDTERM PROBLEMS Due May 3 in class

This is a collection of old midterm questions on Matrices and Linear Transformations, which is the portion of the course that will be covered in the midterm of May 9. The old midterms were of varying lengths of time (1 hour, 2 hours, 3 hours) so the number of these questions that appeared on the midterms also varied.

The solutions to this sampler will be made available on the homework due date, May 3, as always.

We strongly suggest that you also begin working on the next homework, Homework No. 5, which has also been posted and which deals with Group Theory. Since it is due May 10, you can't just start doing it after the midterm. Furthermore, we will begin covering Group Theory shortly and you will find yourself far behind unless you start working on those problems as soon as possible.

1. Let $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3$ be an orthonormal basis set for a 3-dimensional vector space and let

$$\begin{aligned}\hat{\gamma}_1 &= \hat{\alpha}_1 - \hat{\alpha}_2 \\ \hat{\gamma}_2 &= \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3 \\ \hat{\gamma}_3 &= \hat{\alpha}_2 - \hat{\alpha}_3\end{aligned}$$

- a. Is the set $\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3$ an orthonormal basis set?
 - b. Let $T\hat{\alpha}_i = \hat{\gamma}_i$. What is the matrix representing T in the $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3$ basis?
 - c. Let $Q\hat{\gamma}_i = \hat{\alpha}_i$. What matrix represents Q in the $\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3$ basis?
2. **This one is more difficult than we are likely to put on the exam, but it is instructive anyway.** An $n \times n$ Markov matrix is defined to be any real $n \times n$ matrix $A = (a_{ij})$ which satisfies the two properties

$$0 \leq a_{ij} \leq 1$$

and

$$\sum_{j=1}^n a_{ij} = 1 \text{ for each } i.$$

Prove that the product of two Markov matrices is a Markov matrix.

3. Consider the real 3-dimensional space with basis vectors

$$\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

a. Show that the matrix

$$\mathbf{A} = \begin{pmatrix} 1/\sqrt{2} & -1/2 & 1/2 \\ 1/\sqrt{2} & 1/2 & -1/2 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

in this basis represents the operation “rotation of 45° about the \hat{e}_1 axis followed by rotation of 45° about the \hat{e}_3 axis.”

b. Is the matrix \mathbf{A} hermitean?

c. Is the matrix \mathbf{A} unitary?

d. Is the matrix \mathbf{A} singular?

4. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

a. Evaluate $\text{Det } \mathbf{A}$.

b. Evaluate $\text{Tr } \mathbf{A}$.

c. What are the eigenvalues of \mathbf{A} ?

d. What are the eigenvectors of \mathbf{A} ?

e. Evaluate \mathbf{A}^{-1} .

f. Can the matrix \mathbf{A} and the matrix $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ be simultaneously diagonalized?

5. The following are a variety of short disconnected questions. Suppose that \mathbf{B} and \mathbf{C} are $n \times n$ matrices and that \mathbf{X} and \mathbf{Y} are $n \times 1$ matrices.

a. Suppose that \mathbf{B} has eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ and nondegenerate eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Further, suppose that $\mathbf{BC} = \mathbf{CB}$.

i) Do you have enough information to determine the eigenvalues of \mathbf{C} ?

ii) Do you have enough information to determine the eigenvectors of \mathbf{C} ?

b. Suppose that the vectors \mathbf{x} and \mathbf{y} are orthogonal, i.e., that $\mathbf{x}^\dagger \mathbf{y} = 0$. Now consider the new vectors \mathbf{Bx} and \mathbf{By} . What condition must \mathbf{B} satisfy in order to ensure that these new vectors are orthogonal to each other?

c. Suppose that \mathbf{B} is a $s \times 2$ cyclic matrix of order 2 with nondegenerate eigenvalues. What are the eigenvalues of \mathbf{B} ?

d. Suppose that $\mathbf{C}^{-1} \mathbf{BC}$ is diagonal and that \mathbf{B} is hermitean. Is \mathbf{C} necessarily singular or unitary or hermitean? Choose as many as apply and be sure to explain clearly.

6. Suppose that

$$\mathbf{B} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Evaluate $e^{\mathbf{B}}$. Note: there is more than one way to accomplish this.

7. Suppose that we have the basis set

$$\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and that the matrix

$$\mathbf{C} = \frac{1}{\sqrt{s}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

transforms the basis set $\{\hat{e}_i\}$ to a new basis set $\{\hat{e}'_i\}$:

$$\hat{e}'_i = \mathbf{C}\hat{e}_i.$$

- a. The vector \mathbf{v} has components $(3, 5, -1)$ in the $\{\hat{e}_i\}$ basis. What are its components in the $\{\hat{e}'_i\}$ basis?
- b. The transformation that projects all vectors onto \hat{e}_1 is given in the $\{\hat{e}_i\}$ basis as

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

What is the matrix that represents this transformation in the $\{\hat{e}'_i\}$ basis?

8. The matrix \mathbf{A} is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Evaluate the inverse $(\mathbf{1} - \mathbf{A})^{-1}$. There is a **hard** way to do this and a very **easy** way. Try to find the easy way.

9. Evaluate $e^{-2\mathbf{A}^2}$ for

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$