Integration Factor Splitting for the Euler Equations

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Outline

- Semi-Lagrangian Schemes
- Integration Factor Splitting
- Compressible Euler Equations
- Stable Semi-Implicit Formulation
- Schaer Mountain Wave Results
Consider the passive advection problem

\[ \frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = 0, \quad \phi(x, 0) = \phi_0 \]

In the Lagrangian reference frame, equivalent to

\[ \frac{d\phi}{dt} = 0 \]

along characteristics

\[ \frac{dX}{dt} = \mathbf{v} \]
Semi-Lagrangian

\[ t^n + \Delta t \]

\[ \frac{1}{a} \]

\[ x_j \quad x_i \]

\[ t^n \]
Semi-Lagrangian

\[ t^n + \Delta t \]

\[ \frac{1}{a} \]

\[ x_j \]

\[ x_i' \]
Semi-Lagrangian

\[ t^n + \Delta t \]

\[ \frac{1}{a} \]

\[ x_j \rightarrow x_i \]

\[ t^n \]
Integration Factor Splitting

\[ t^n + \Delta t \]

\[ \frac{1}{a} \]

\[ x_j \quad x \]
Integration Factor Splitting

\[ t^n + \Delta t \]

\[ \frac{1}{a} \]

\[ x_j \quad x \]
Integration Factor Splitting

\[ t^n + \Delta t \]

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\[ x_j \quad x \]

\[ t^n \]
Integration Factor Splitting

\[ t^n + \Delta t \]

\[ t^n \]

\[ \frac{1}{a} \]

\[ x_j \]

\[ x \]
Integrating Factor Splitting

\[
\frac{du}{dt} = S(u) + F(u), \quad t \in [0, T]
\]

Integrating factor \( Q^*_S(t) \),

\[
\frac{d}{dt} \left[ Q^*_S(t) \cdot u \right] = Q^*_S(t) \cdot F(u).
\]

To compute \( Q^*_S(t) \cdot u \) integrate

\[
\frac{dv^{(t^*,t)}(s)}{ds} = S(v^{(t^*,t)}), \quad 0 \leq s \leq t^* - t
\]

with initial condition \( v^{(t^*,t)}(0) = u(t) \)
Linear Case

When $S(u) = Au$ and $A$ is time-independent,

$$Q_{S}^{t^*}(t) = e^{A(t^*-t)}$$

Integration over $[t, t^*]$ yields

$$u(t^*) = e^{A(t^*-t)}u(t) + \int_{t}^{t^*} e^{A(t^*-s)}F(u(s)) \, ds.$$  

and $Q_{S}^{t^*}(t^*) = I$
If \( v(t^*,t)(s) \) is the solution of

\[
\frac{d}{ds} v(t^*,t)(s) = S(v(t^*,t)(s)), \quad 0 \leq s \leq t^* - t
\]

with initial condition \( v(t^*,t)(0) = u(t) \), then

\[
Q_S(t^*) \cdot u(t) = v(t^*,t)(t^* - t)
\]
Su = −u · ∇u is the advection operator
X(x, t) is the Lagrangian trajectory
u(X(x, t), t) is the velocity

\[ \frac{d}{dt} u(X(x, t), t) = \frac{\partial u(x, t)}{\partial t} - S(u(x, t)) \]

and \( X(x, t) \) satisfies

\[ \frac{d}{dt} X(x, t) = u(X(x, t), t). \]

St-Cyr and Thomas (2004)
Theorem

If $X(x, t^{n-q})$ is the solution of

$$\frac{d}{dt} X(x, t) = u(X(x, t), t), \quad t \in [t^{n-q}, t^n]$$

with $X(x, t^n) = x$ and $v^{(t^n, t^{n-q})}(s)$ is the solution of (1) with initial condition $v^{(t^n, t^{n-q})}(0) = u(x, t^{n-q})$ then

$$u(X(x, t^{n-q}), t^{n-q}) = v^{(t^n, t^{n-q})}(t^n - t^{n-q}).$$

$$u(X(x, t^{n-q}), t^{n-q}) = Q_{t^n}^{t^{n-q}}(s) \cdot u(t^{n-q}) = \tilde{u}(x, t^{n-q})$$
Advection

Time-splitting implies solving the IVP:

\[ \frac{\partial \tilde{\phi}}{\partial s} + v \cdot \nabla \tilde{\phi} = 0, \quad \tilde{\phi}(x, t^{n-q}) = \phi(x, t^{n-q}) \]

Direct space-time discretization in 1-D:

\[ \phi_{j}^{n+1} = \phi_{j}^{n} + \frac{\Delta t v_j}{\Delta x} (\phi_{j-1}^{n} - \phi_{j}^{n}) \]

\[ x_j^* = x_j - \Delta t v(x_j, t^n) \]

First-order upwind – dissipative
Advection

Upwind is based on Lagrange interpolation:

\[ \phi^{n+1}_j = L_{j-1}(x_j - x^*_j)\phi^n_{j-1} + L_j(x_j - x^*_j)\phi^n_j \]

Space-time truncation error \( \Delta t \rho^n_j \)
\( \rho^n_j \) from the interpolation error

Quadratic interpolation: Lax-Wendroff

\[ \phi^{n+1}_j = L_{j-1}(x_j - x^*_j)\phi^n_{j-1} + L_j(x_j - x^*_j)\phi^n_j + L_{j+1}(x_j - x^*_j)\phi^n_{j+1} \]
Method-of-lines (MOL) discretizations
Use one-step RK-3 or RK-4 for stability

\[ \frac{\partial \tilde{\phi}_j}{\partial s} + \mathbf{v}^n \cdot \nabla \tilde{\phi}_j = 0 \]

Problem: accuracy of the advecting wind \( \mathbf{v} \) ?
Time-Split Leap-Frog

Model variable $u$

$$\frac{du}{dt} = S(u) + F(u)$$

$$\frac{u^{n+1} - Q_t^{n+1} u^{n-1}}{2\Delta t} = \frac{1}{2} \left( F^{n+1} + Q_S^{n+1} F^{n-1} \right)$$

Implicit Solver

$$u^{n+1} - \Delta t F^{n+1} = \tilde{u}^{n-1} - \Delta t \tilde{F}^{n-1}$$
Assume constant matrices $S$ and $F$

$$\frac{du}{dt} = Su + Fu$$

Splitting error is the difference between

$$u(t^{n+k}) = e^{k\Delta t (S+F)}u(t^n)$$

and

$$u(t^{n+k}) = e^{k\Delta t F} e^{k\Delta t S} u(t^n)$$
Splitting Error

Alternative: Strang splitting

\[ u(t^{n+1}) = e^{\Delta t S/2} e^{\Delta t F} e^{\Delta t S/2} u(t^n) \]

Splitting error same order as the scheme for \( F(u) \)
2-D Euler equations

\[
\frac{du}{dt} + RT \frac{\partial q}{\partial x} = 0
\]

\[
\frac{dw}{dt} + RT \frac{\partial q}{\partial z} + g = 0
\]

\[
\frac{dT}{dt} - \frac{RT}{c_p} \frac{dq}{dt} = 0
\]

\[
\frac{c_v}{c_p} \frac{dq}{dt} + D = 0
\]

\[
T' = T - T_*, \quad q' = q - q_*, \quad \partial q_*/\partial z = -g/RT_*
\]
Hydrostatic Basic State

\[ b = gT'/T_*, \quad P = RT_* q', \quad N_*^2 = g^2/c_p T_* = g\gamma_* \]
\[ c_*^2 = (c_p/c_v)RT_* \]

\[
\frac{du}{dt} + \frac{\partial P}{\partial x} = -\frac{b}{g} \frac{\partial P}{\partial x}
\]
\[
\frac{dw}{dt} + \frac{\partial P}{\partial z} - b = -\frac{b}{g} \frac{\partial P}{\partial z}
\]
\[
\frac{d}{dt}(b - \gamma_* P) + N_*^2 w = -\frac{R}{c_v} bD
\]
\[
\frac{1}{c_*^2} \left[ \frac{dP}{dt} - gw \right] + D = 0
\]
\( \delta X = X^{n+1} - X^{n-1} \), \( \overline{X} = \left[ (1 + \varepsilon)X^{n+1} + (1 - \varepsilon)X^{n-1} \right] / 2 \)

\[
\frac{\delta u}{2\Delta t} + \frac{\partial \overline{P}}{\partial x} = -\frac{b}{g} \frac{\partial P}{\partial x}
\]

\[
\frac{\delta w}{2\Delta t} + \frac{\partial \overline{P}}{\partial z} - \overline{b} = -\frac{b}{g} \frac{\partial P}{\partial z}
\]

\[
\frac{\delta (b - \gamma^*P)}{2\Delta t} + N^2 w = -\frac{R}{c_v} bD
\]

\[
\frac{1}{c_*^2} \left[ \frac{\delta P}{2\Delta t} - g\overline{w} \right] + \overline{D} = 0
\]

Bénard, MWR (2003), (2004): Unstable when \( \varepsilon = 0 \)
Stable Scheme

\[ b = gT'/T, \quad P = RT_* q', \quad \alpha_0 = T'/T_*, \quad \theta_0 = 1 + \alpha_0 \]
\[ N_0^2 = N^2_*/(1 + \alpha_0), \quad c_0^2 = c_*^2(1 + \alpha_0) \]

\[ \frac{\delta u}{2\Delta t} + \theta_0 \frac{\partial P}{\partial x} = 0 \]
\[ \frac{\delta w}{2\Delta t} + \theta_0 \frac{\partial P}{\partial z} - b = 0 \]
\[ \theta_0 \frac{\delta b}{2\Delta t} - \gamma^* \frac{\delta P}{2\Delta t} + N_0^2 w = (N_0^2 - N^2_*) w \]
\[ \frac{1}{c_*^2} \frac{\delta P}{2\Delta t} + D - \frac{g}{c_*^2} w = - \left[ \frac{g}{c_0^2} - \frac{g}{c_*^2} \right] \]

Girard and Desgagné (2005)